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THE ALMOST PERIODIC LYAPUNOV PROBLEM†

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The necessary and sufficient conditions for optimum generalized controls (measures) in the almost periodic (a.p.) Lyapunov problem are presented. Some results are also presented regarding the legitimacy of extension and the needle-sharp variation of an almost-periodic control is constructed. Extension procedures for optimal control problems have been considered, e.g. in [1–3], and in a game-theoretic context in [4, 5].

1. LET R^n be a Euclidean n-space, $|x|$ the norm of an element $x\in R^n$ and U a compact subset of R^n. Let $S_1(R,X)$ ($l>0$, $X\subseteq R^n$) denote the collection of all almost periodic (a.p.) functions in Stepanov's sense (henceforth we shall say simply "a.p. function"). We recall [6] that a locally integrable function f: $R\to X$ belongs to $S_1(R,X)$ if, for any $\varepsilon>0$, its set

$$E_l(f,\varepsilon)\doteq\left\{\tau\in R:\sup_{t\in R}\frac{1}{l}\int_t^{t+l}|f(s+\tau)-f(s)|\,ds<\varepsilon\right\}$$

($E(f,\varepsilon)\doteq E_1(f,\varepsilon)$) of ε-translation numbers is relatively dense. For every $l>0$,

$$S(R,X)\doteq S_1(R,X)\subset S_l(R,X),\quad S_l(R,X)\subset S(R,X)$$

so we may confine our attention to $S(R,X)$. In addition, every function $f\in S(R,X)$ may be associated with its Fourier series, which is conveniently expressed in complex form:

$$f(t)\sim\sum_\lambda F(\lambda)e^{i\lambda t},\qquad F(\lambda)\doteq M\{f(t)e^{-i\lambda t}\}\doteq\lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)e^{-i\lambda t}\,dt$$

where the set $\Lambda(f)\doteq\{\lambda\in R:|F(\lambda)|>0\}$ of Fourier exponents of f (its spectrum) is at most denumerable.

A set $F\subset S(R,X)$ is uniformly a.p. if, for any $\varepsilon>0$, the set

$$\bigcap_{f\in F}E(f,\varepsilon)$$

is relatively dense.

Throughout this paper $\operatorname{Mod}(\Delta)$ will denote the module of a set $\Delta\subseteq R$, i.e. the smallest additive group containing Δ, and, if $f\in S(R,X)$, then $\operatorname{Mod}(f)\doteq\operatorname{Mod}(\Lambda(f))$ will denote the modulus of the function f.

Let us assume that the following elements are given: the functions $f_k\in S(R,R)$, $g_k\in C(U,X)$ ($k=0,\ldots,m',\ldots,m$), a set $\Delta\subseteq R$ and a subset $I(U)\subset S(R,X)$, called the set of (ordinary) controls, of functions $u(\cdot)$ such that $\operatorname{Mod}(u)\subseteq\operatorname{Mod}(\Delta)$. The extremal problem

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$$J_0(u(\cdot)) \to \inf,\ u(\cdot) \in I(U)$$
$$J_k(u(\cdot)) \leqslant 0,\ k = 1, 2, \ldots, m',\ J_k(u(\cdot)) = 0,\ k = m' + 1, \tag{1.1}$$

$$m' + 2, \ldots, m \qquad (J_k(u(\cdot)) \doteq M\{f_k(t)\, g_k(u(t))\}) \tag{1.2}$$

will be called the a.p. Lyapunov problem (cf. the Lyapunov problem in [7]) and the set D_1 of controls $u(\cdot) \in I(U)$ that satisfy conditions (1.2) will be called the set of (ordinary) admissible controls for the problem.

Remark 1.1. The results derived below can be extended to a broader class of problems with functionals $J_k(u(\cdot)) \doteq M\{\varphi_k(t, u, (t))\}$, where $\varphi_k(t, u)$ is a uniformly a.p. function of t with respect to $u \in U$.

In order to formulate and investigate a convexified problem for (1.1), (1.2), we need a few propositions, to which we devote the next section.

2. Let frm (U) denote the linear space of Radon measures on R^n whose supports are contained in U, and rpm(U) the subset of frm(U) formed by the Radon probability measures: $N \doteq N(R, \mathrm{frm}(U))$ the set of Lebesgue-measurable maps $\mu: R \to \mathrm{frm}(R)$ such that $\|\mu\| \doteq \mathrm{essup}_{t \in R} |\mu(t)|(U) < \infty$ [$|\mu(t)|(U)$ is the variation of $\mu(t)$], N_1 the subset of N formed by the measurable maps $\mu: R \to \mathrm{rpm}(U)$. Further, let $B \doteq B(R \times U, R^n)$ denote the normed linear space of all functions $\varphi: R \times U \to R^n$ such that the map $t \to \varphi(t, u)$, $u \in U$, is measurable, $\varphi(t, \cdot) \in C(U) \doteq C(U, R^n)$ for a.e. $t \in R$ and there exists a function $\psi_\varphi \in L_1(R, R)$ such that $\max_{u \in U} |\varphi(t, u)| \leqslant \psi_\varphi(t)$ for a.e. $t \in R$. By a slight modification of the proof of the Dunford–Pettis Theorem [2] one can show that $N \cong B_1{}^*$, where $B_1 \doteq B(R \times U, R)$ and the map $\|\cdot\|_w: N \to R$, defined for $\mu \in N$ by

$$\|\mu\|_w \doteq \sum_{j=1}^{\infty} \frac{2^{-j}}{1 + \|\varphi_j\|_B} \left| \int_R \langle \mu(t), \varphi_j(t, u) \rangle\, dt \right|$$

where $\{\varphi_1, \varphi_2, \ldots\}$ is a denumerable dense set of functions in B

$$\|\varphi_j\|_B \doteq \int_R \max_{u \in U} |\varphi(t, u)|\, dt, \quad \langle \mu(t), \quad \varphi_j(t, u) \rangle \doteq \int_U \varphi_j(t, u)\, \mu(t)(du)$$

defines a weak norm in N. The space $(N, \|\cdot\|_w)$ is separable, the set $N_1 \subset (N, \|\cdot\|_w)$ is convex and compact and if $\mu_j, \mu \in N_1$, then $\lim_{j \to \infty} \|\mu_j - \mu\|_w = 0$ (we write $\mu_j \to \mu$ as $j \to \infty$) if and only if

$$\lim_{j \to \infty} \int_R \langle \mu(t) - \mu_j(t), \varphi(t, u) \rangle\, dt = 0$$

for any $\varphi \in B$.

Definition 2.1. A map $\mu \in N$ is said to be a.p. if, for any function $g \in C(U)$, the map $t \to \langle \mu(t), g(u) \rangle$ belongs to $S(R, R^n)$.

We denote the set of all a.p. maps $\mu \in N$ ($\mu \in N_1$) by APM (APM$_1$) (as the structure of APM has been investigated in some detail,† we will limit ourselves here to a concise exposition of the results necessary for our purposes). We identify each function $u(\cdot) \in S(R, U)$ with $\delta_{u(\cdot)} \in \mathrm{APM}_1$ [where

† IVANOV A. G., Measure-valued almost-periodic functions. Preprint, FTI Ural. Otd. Akad. Nauk SSSR, Sverdlovsk, 1990.

$\delta_{u(t)}$ is the Dirac measure with support at the point $u(t) \in U]$, thus embedding $S(R, U)$ (up to an algebraic isomorphism) in APM_1 and hence also in APM.

We will now define Fourier series for elements of APM. If $\mu \in \mathrm{APM}$, then by definition the map $t \to \langle \mu(t), g(u) \rangle$ is an element of $S(R, R^n)$ for any $g \in C(U)$. Let

$$\langle \mu\ (t),\ g\ (u) \rangle \sim \sum_{\lambda} A_\mu\ [g,\ \lambda]\ e^{i\lambda t},\ A_\mu\ [g,\ \lambda] \doteq M\ \{\langle \mu\ (t),\ g\ (u) \rangle\ e^{-i\lambda t}\}$$

and let $\Lambda(\mu, g)$ be the spectrum of the a.p. map $t \to \langle \mu(t), g(u) \rangle$. Using Riesz's Representation Theorem, the linearity of the map $g \to A_\mu[g, \lambda]$ and the condition $\|\mu\| < \infty$, one can show that for every $\lambda \in R$ there exists a measure $\nu_\lambda \in \mathrm{frm}(U)$ such that $\langle \nu_\lambda, g(u) \rangle = A_\mu[g, \lambda]$ for all $g \in C(U)$.

Let $\Lambda(\mu) \doteq \{\lambda \in R : |\nu_\lambda|(U) > 0\}$ and let $\{c_1, c_2, \ldots\} \doteq C_\infty(U)$ be a denumerable set of continuous functions, dense in $C(U)$.

Theorem 2.1. If $\mu \in \mathrm{APM}$, then

$$\Lambda\ (\mu) = \bigcup_{j=1}^{\infty} \Lambda\ (\mu,\ c_j)$$

and so the set $\Lambda(\mu)$ is at most denumerable.

Hence the following definition is legitimate.

Definition 2.2. If $\mu \in \mathrm{APM}$, the at most denumerable set of real numbers $\Lambda(\mu)$ is called the set of Fourier exponents of μ; the measures $\nu_\lambda \in \mathrm{frm}(U)$ $[\nu_\lambda = 0$ if $\lambda \notin \Lambda(\mu)]$ such that $\langle \nu_\lambda, g(u) \rangle = A_\mu[g, \lambda]$ for all $g \in C(U)$ are called the Fourier coefficients of μ. Finally, the measure-valued series $\Sigma_\lambda \nu_\lambda e^{i\lambda t}$ is called the Fourier series of μ and the set $\mathrm{Mod}(\mu) \doteq \mathrm{Mod}(\Lambda(\mu))$ its modulus.

Theorem 2.2. For any $\mu \in \mathrm{APM}_1$ there exists a sequence of functions $\{u_j\}_{j=1}^{\infty} \subset S(R, U)$ such that $\mathrm{Mod}(u_j) \subset \mathrm{Mod}(\mu)$, $j = 1, 2, \ldots$; $\delta_{u_j(t)} \to \mu(t)$ as $j \to \infty$. In addition

$$\lim_{t \to \infty} M\ \{f\ (t)\ g\ (u_j\ (t))\} = M\ \{f\ (t)\ \langle \mu\ (t),\ g\ (u) \rangle\}$$

for any function $f \in S(R, R)$ and $g \in C(U)$.

3. Henceforth P will be the subset of APM_1 consisting of all maps μ such that $\mathrm{Mod}(\mu) \subset \mathrm{Mod}(\Delta)$; this will be called the set of (generalized) controls.

Definition 3.1. The extremal problem

$$J_0\ (\mu\ (\cdot)) \to \inf,\ \mu\ (\cdot) \in P \tag{3.1}$$

$$J_k\ (\mu\ (\cdot)) \leqslant 0,\ k = 1, 2, \ldots, m',\ J_k\ (\mu\ (\cdot)) = 0,\ k = m' + 1,\ m' + 2, \ldots, m \tag{3.2}$$

$$(J_k\ (\mu\ (\cdot)) \doteq M\ \{f_k\ (t)\ \langle \mu\ (t),\ g_k\ (u) \rangle\},\ k = 0,\ 1,\ \ldots,\ m)$$

is called the convexified a.p. Lyapunov problem for problem (1.1), (1.2), and the set $D_2 \subset P$ of generalized controls satisying conditions (3.2) is the set of (generalized) admissible controls.

Using Theorem 2.2, one can prove the following theorem.

Theorem 3.1. For any solution μ of problem (3.1), (3.2) there exists a sequence of admissible controls $\{u_j(\cdot)\}_{j=1}^{\infty} \subset D_1$ of problem (1.1), (1.2) such that $J_0(u_j(\cdot)) \to J_0(\mu(\cdot))$ as $j \to \infty$. We call the function

$$L : \operatorname{rpm}(U) \times R^{m*} \times R \to R, \quad L(\nu, \lambda, \lambda_0) \doteq \sum_{k=0}^{m} \lambda_k J_k(\nu)$$

the Lagrange function of problem (3.1), (3.2).

Theorem 3.2. A necessary condition for $\mu \in D_2$ to be a solution of problem (3.1), (3.2) is that there exist a number $\lambda_0 \geqslant 0$ and a vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in R^{m*}$, not simultaneously zero, such that

1. For a.e. $t \in R$ the following minimum principle holds:

$$\min_{\nu \in \operatorname{rpm}(U)} \left\langle \nu, \sum_{k=0}^{m} \lambda_k f_k(t) g_k(u) \right\rangle = \left\langle \mu(t), \sum_{k=0}^{m} \lambda_k f_k(t) g_k(u) \right\rangle$$

2. $\lambda_k \geqslant 0$. $\lambda_k J_k(\mu(\cdot)) = 0$, $k = 1, \ldots, m'$.

If $\mu \in D_2$ and there exist a number $\lambda_0 > 0$ and a vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in R^{m*}$, such that both these conditions hold, then μ is a solution of problem (3.1), (3.2).

The proof of Theorem 3.2 will be given at the end of the paper, since it relies on the notion of the needle-shaped variation of an a.p. generalized control and on the elementary a.p. Lyapunov problem.

4. We shall say that a sequence of functions $\{f_m\}_{m \in Z}$ in $L_1([0, a], X)$ $(a > 0)$ is a.p. if, for any $\varepsilon > 0$, its set

$$\left\{ l \in Z : \sup_{m \in Z} \int_0^a |f_{m+l}(t) - f_m(t)|\, dt < \varepsilon \right\}$$

of ε-translation numbers is relatively dense.

Lemma 4.1. $f \in S(R, X)$ if and only if the sequence of functions $\{f_m\}_{m \in Z}$ in $L_1([a, 0], X)$, where $f_m(t) \doteq f(t + ma)$ for $t \in [0, a]$, is a.p.

Definition 4.1. Let $N_1[0, a]$ be the set of measurable maps $\mu : [0, a] \to \operatorname{rpm}(U)$. A sequence $\{\mu_m\}_{m \in Z}$ of elements of $N_1[0, a]$ is said to be a.p., if, for any function $g \in C(U)$, the sequence $\{\langle \mu_m(\cdot), g(u) \rangle\}_{m \in Z} \subset L_1([0, a], R^n)$ is a.p.

From Lemma 4.1 we can derive the following lemma.

Lemma 4.2. $\mu \in \mathrm{APM}$ if and only if the sequence $\{\mu_m\}_{m \in Z}$, where $\mu_m(t) \doteq \mu(t + ma)$ for $t \in [0, a]$, in $N_1[0, a]$ is a.p.

Let $\mu \in \mathrm{APM}_1$ and let $\nu \in N_1$ be an a-periodic map. We define a measure for $t \in [0, a]$ and $m \in Z$:

$$\mu_m(t, \alpha) \doteq \begin{cases} \mu_m(t) \doteq \mu(t + ma), & t \in [0, a] \setminus [\vartheta, \vartheta + \alpha) \\ \nu(t), & t \in [\vartheta, \vartheta + \alpha) \end{cases}$$

where $\vartheta \in (0, a)$ and $\alpha > 0$ is such that $[\vartheta, \vartheta + \alpha) \subset [0, a]$. By Lemma 4.2, the sequence $\{\mu_m\}_{m \in Z} \subset N_1[0, a]$ is a.p. Taking this into account, one can show that the sequence $\{\mu_m(\cdot, \alpha)\}_{m \in Z} \subset N_1[0, a]$ is also a.p. Now consider the map $\mu(\cdot, \alpha) \in N_1$ whose restriction to $[ma, (m+1)a]$ is $\mu_m(t, \alpha)$. By Lemma 4.2, $\mu(\cdot, \alpha) \in \mathrm{APM}_1$. Clearly

$$\mu(t,\alpha)=\begin{cases}\mu(t), & t\in\bigcup_{m\in Z}[ma,(m+1)a]\setminus T_{m,\alpha,\vartheta}\\ \nu(t), & t\in\bigcup_{m\in Z}T_{m,\alpha,\vartheta}\doteq\bigcup_{m\in Z}[ma+\vartheta,ma+\vartheta+\alpha]\end{cases}\tag{4.1}$$

The map $\mu(\cdot,\alpha)\in\mathrm{APM}_1$, defined by (4.1) will be called a needle-shaped variation of $\mu(\cdot)\in\mathrm{APM}_1$. We note that the set $\{\mu(\cdot,\alpha)\}_{0<\alpha<a-\vartheta}\subset\mathrm{APM}_1$ is uniformly a.p. [i.e. for any function $g\in C_\infty(U)$ the set $\{\langle\mu(\cdot,\alpha),g(u)\rangle\}_{0<\alpha\leq a-\vartheta}\subset S(R,R^n)$ is uniformly a.p.].

Theorem 4.1. If $\mu\in\mathrm{APM}_1$, then its needle-shaped variation $\mu(\cdot,\alpha)$ is also in APM_1, and if $2\pi/a\Lambda(\mu)$, then

$$\mathrm{Mod}\,(\mu\,(\cdot,\alpha))\subset\mathrm{Mod}\,(\mu)\tag{4.2}$$

Proof. The inclusion $\mu(\cdot,\alpha)\in\mathrm{APM}_1$ was proved above. To determine $\Lambda(\mu(\cdot,\alpha))$, let

$$\mu\,(t,\alpha)\sim\sum_\lambda\nu_\lambda(\alpha)\,e^{i\lambda t},\quad \nu_\lambda(\alpha)\in\mathrm{frm}\,(U)$$

[$\nu_\lambda(\alpha)=0$ if $\lambda\notin\Lambda(\mu(\cdot,\alpha))$, $\langle\nu_\lambda(\alpha),g(u)\rangle=M\{\langle\mu(t,\alpha),g(u)\rangle e^{-i\lambda t}\}$ for all $g\in C(U)$]. We then deduce from (4.1) that

$$\langle\nu_\lambda(\alpha),g(u)\rangle=\lim_{k\to\infty}\frac{1}{ka}\sum_{m=0}^{k-1}\int_{T_{m,\alpha,\vartheta}}\langle\nu(t),g(u)\rangle\,e^{-i\lambda t}\,dt+I$$

$$I\doteq\lim_{k\to\infty}\frac{1}{ka}\sum_{m=0}^{k-1}\left(\int_{ma}^{ma+\vartheta}\langle\mu(t),g(u)\rangle\,e^{-i\lambda t}\,dt+\int_{ma+\vartheta+\alpha}^{(m+1)a}\langle\mu(t),g(u)\rangle\,e^{-i\lambda t}\,dt\right)$$

Consider a a-periodic map $\mu^{(1)}\in N$ and the map $\mu^{(2)}\in\mathrm{APM}$ defined on $[ma,(m+1)a]$ by

$$\mu^{(1)}(t)\doteq\chi_{T_{m,\alpha,\vartheta}}(t)\,\nu(t)+(1-\chi_{T_{m,\alpha,\vartheta}}(t))\,\eta$$

$$\mu^{(2)}(t)\doteq-\chi_{T_{m,\alpha,\vartheta}}(t)\,\mu(t)+(1-\chi_{T_{m,\alpha,\vartheta}}(t))\,\eta$$

where $(\chi_{T_{m,\alpha,\vartheta}}(\cdot))$ is the characteristic function of $T_{m,\alpha,\vartheta}$. If

$$\mu(t)\sim\sum_\lambda\nu_\lambda e^{i\lambda t},\quad \mu^{(2)}(t)\sim\sum_\lambda\nu_\lambda^{(2)}e^{i\lambda t}$$

then $I=\langle\nu_\lambda+\nu_\lambda^{(2)},g(u)\rangle$. Therefore, for any function $g\in C(U)$

$$\langle\nu_\lambda(\alpha),\ g(u)\rangle=\langle\nu_\lambda+\nu_\lambda^{(2)},\ g(u)\rangle+M\,\{\langle\mu^{(1)}(t),\ g(u)\rangle\,e^{-i\lambda t}\}$$

Hence, using the fact that $\Lambda(\mu^{(1)},g)=(2\pi/a)Z$ and Theorem 2.1, we see that

$$\Lambda\,(\mu(\cdot,\ \alpha))=\Lambda\,(\mu)\cap\Lambda\,(\mu^{(2)})\cap(2\pi/a)\,Z$$

and since $2\pi/a\in Z$

$$\mathrm{Mod}\,(\mu\,(\cdot,\ \alpha))\subset\mathrm{Mod}\,(\mu)\cup\mathrm{Mod}\,(\mu^{(2)})$$

Now, it follows from the definition of $\mu^{(2)}$ that for every function $g\in C(U)$ and any $\varepsilon>0$ there exists $\delta>0$ such that

$$E\,(\langle\mu\,(\cdot),\ g\,(u)\rangle,\ \delta)\subset E\,(\langle\mu^{(2)}\,(\cdot),\ g\,(u)\rangle,\ \varepsilon)$$

Consequently, by Favard's Theorem [6, p. 126],

$$\mathrm{Mod}\,(\langle \mu^{(2)}(\cdot),\ g(u)\rangle) \subseteq \mathrm{Mod}\,(\langle \mu(\cdot),\ g(u)\rangle)$$

for any function $g \in C(U)$, which in turn implies the inclusion $\mathrm{Mod}(\mu^2) \subset \mathrm{Mod}(\mu)$. This proves (4.2).

Definition 4.2. The extremal problem

$$I_0(\mu(\cdot)) \doteq M\left\{\left\langle \mu(t), \sum_{k=0}^{m} f_k(t)\, g_k(u)\right\rangle\right\} \to \inf, \mu(\cdot) \in P \tag{4.3}$$

is called an elementary a.p. Lyapunov problem.

Theorem 4.2. A generalized control $\mu(\cdot) \in P$ is a solution of problem (4.3) if and only if, for a.e. $t \in R$

$$\min_{\nu \in \mathrm{rpm}(U)} \left\langle \nu, \sum_{k=0}^{m} f_k(t)\, g_k(u)\right\rangle = \left\langle \mu(t), \sum_{k=0}^{m} f_k(t)\, g_k(u)\right\rangle \tag{4.4}$$

Proof. The sufficiency is obvious. To prove the necessity, we may assume without loss of generality that all the functions $f_k \colon R \to R$ are a.p. in Bohr's sense. This follows from the fact that if $f_k \in S(R, R)$ and

$$f_k^{(h)}(t) \doteq \frac{1}{h} \int_t^{t+h} f_k(s)\, ds, \qquad h > 0,\, t \in R$$

then [6, p. 207] for every $l > 0$

$$\lim_{h \downarrow 0} \left(\sup_{t \in R} \frac{1}{l} \int_t^{t+l} |f_k(s) - f_k^{(h)}(s)|\, ds\right) = 0$$

Consider the problem

$$I(\mu(\cdot)) \doteq M\{\langle \mu(t),\ \varphi(t, u)\rangle\} \to \sup,\ \mu(\cdot) \in P \tag{4.5}$$

$$\varphi(t, u) \doteq \sum_{k=0}^{m} (\|f_k\|_{C(R,R)} \|g_k\|_{C(U,R)} - f_k'(t)\, g_k(u))$$

Clearly, $\mu(\cdot) \in P$ is a solution of problem (4.3) if and only if it is a solution of problem (4.5), and condition (4.4) may be written equivalently as

$$\max_{\nu \in \mathrm{rpm}(U)} \langle \nu,\ \varphi(t, u)\rangle = \langle \mu(t),\ \varphi(t, u)\rangle \tag{4.6}$$

Thus, we have to prove that if $\mu(\cdot) \in P$ is a solution of problem (4.5), then condition (4.6) is satisfied for almost all $t \in R$. Choose $a > 0$ so that $2\pi/a \in \Lambda(\mu)$. Suppose that condition (4.6) fails to hold. Then there exist $\alpha > 0$, $\vartheta \in R$ (to fix our ideas, we assume that $[\vartheta, \vartheta + \alpha) \in [0, a]$) and a measurable function $u_0 \colon [\vartheta, \vartheta + \alpha) \to U$ such that $\varphi(t, u_0 t)) - \langle \mu(t), \varphi(t, u)\rangle > 0$ for all $t \in [\vartheta, \vartheta + \alpha)$ using the fact that $(t, u) \in R \times U$ for $\varphi(t, u) \geqslant 0$, that all the measures here belong to $\mathrm{rpm}(U)$ for all t and also using a theorem of Filippov [2]. Extend the function $u_0 \colon [\vartheta, \vartheta + \alpha) \to U$ first by defining it as zero on $[0, a]\setminus[\vartheta, \vartheta+\alpha)$, and then continue it to an a-periodic function on R. Denote the extended function by v_0. For every $m\in Z$ and $t\in T_{m,\alpha,\vartheta}$ the problem

$$\langle \nu, \varphi(t, u)\rangle \to \sup, \quad \nu\in \mathrm{rpm}(U)$$

has a solution. Hence for any $m\in Z$ there exists a measurable function $u_m: T_{m,\alpha,\vartheta}\to U$, such that $\langle\mu(t), \varphi(t, u)\rangle \leqslant \varphi(t, u_m(t))$ for $t\in T_{m,\alpha,\vartheta}$. We now again extend the function $u_m: T_{m,\alpha,\vartheta}\to U$, by defining it as zero on $[ma, (m+1)a]\setminus T_{m,\alpha,\vartheta}$ and then continuing it a-periodically to all of R. Denote the extended function by v_m. Now consider the a-periodic function $u(t) \doteq \sup_m v_m(t)$ and, letting $\nu(t)\doteq\delta_{u(t)}$, consider the needle-shaped variation $\mu(\cdot,\alpha)$ for $\mu(\cdot)$ defined by formula (4.1). By Theorem 4.1 and the inclusion (4.2), $\mu(\cdot,\alpha)\in P$. Therefore $I(\mu(\cdot,\alpha))\leqslant I(\mu(\cdot))$. On the other hand, $\langle\mu(t,\alpha)-\mu(t), \varphi(t, u)\rangle\geqslant 0$ for $t\in R$, and so, the function

$$t\to F(t) \doteq \int_t^{t+\alpha} \langle\mu(s,\alpha)-\mu(s), \varphi(s, u)\rangle\, ds, \quad t\in R$$

which is a.p. in Bohr's sense, is non-negative. But since $F(\vartheta)\doteq\gamma>0$, it follows from a known property of a.p. function in Bohr's sense [6, p. 46] that there exists $q>0$ ($qa>\alpha$) such that every interval $[mqa, (m+1)aq]$ contains a point t_m at which $F(t_m)>\gamma/3$. Therefore

$$I(\mu(\cdot,\alpha)) - I(\mu(\cdot)) = \lim_{k\to\infty}\frac{1}{kqa}\sum_{m=0}^{k-1}\int_{mqa}^{(m+1)qa}\langle\mu(s,\alpha)-\mu(s), \varphi(s, u)\rangle\, ds \geqslant$$
$$\geqslant \lim_{k\to\infty}\frac{1}{kqa}\sum_{m=0}^{k-1} F(t_m) > \gamma/3qa$$

i.e. $I(\mu(\cdot,\alpha))>I(\mu(\cdot))$. This contradiction completes the proof of Theorem 4.2.

Remark 4.1. Theorem 2.2 implies a result analogous to Theorem 3.1: if $\mu(\cdot)\in P$ is a solution of problem (4.3), then there exists a sequence $\{u_j\}_{j=1}^{\infty}\subset I(U)$ such that $\delta u_j(t)\to\mu(t)$ as $j\to\infty$.

Proof of Theorem 3.2. Let $\mu(\cdot)\in D_2$ be a solution of problem (3.1), (3.2). We may assume without loss of generality that $J_0(\mu(\cdot))=0$. Consider the set $G\doteq\{\alpha=(\alpha_0,\ldots,\alpha_m)$: there exists $\nu(\cdot)\in P$ such that $J_0(\nu(\cdot))<\alpha_0$, $J_k(\nu(\cdot))\leqslant\alpha_k$, $k=1, 2, \ldots, m'$, $J_k(\nu(\cdot))=\alpha_k$, $k=m'+1, m'+2, \ldots, m\}$. Since the maps $\mu(\cdot)\to J_k(\mu(\cdot))$, $k=0, 1, \ldots, m$ are linear, it follows that G is a convex subset of $R^{(1+m)*}$. In addition G does not contain the origin [otherwise $J_0(\mu_0(\cdot))<0$, $J_k(\mu_0(\cdot))\leqslant 0$, $k=1, 2, \ldots, m'$, $J_k(\mu_0(\cdot))=0$, $k=m'+1, m'+2, \ldots, m$ for some $\mu_0(\cdot)\in P$]. Applying the finite-dimensional separation theorem [7, p. 53], we find numbers λ_k ($k=0, 1, \ldots, m$), not all zero, such that $\lambda_0\alpha_0+\lambda_1\alpha_1+\ldots+\lambda_m\alpha_m\geqslant 0$ for all $\alpha\in G$. Since $\alpha=(1, 0, \ldots, 0)\in G$, it follows that $\lambda_0\geqslant 0$. Now, following the scheme of the proof for the standard Lyapunov problem [7], we prove the validity of condition 2 of Theorem 3.2 and the minimum principle for the Lagrange function of problem (3.1), (3.2). Theorem 4.2 may then be used to establish the minimum principle for problem (3.1), (3.2).

The sufficiency can be verified directly.

Remark 4.2. We have devoted no attention to the question of the existence of solutions, as it involves a compactification procedure for the space APM_1—a topic which merits a special discussion.

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CONTROL OF A PREDATOR–PREY SYSTEM WITH INTRASPECIES COMPETITION†

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The time-optimal control problem for a predator–prey system with intraspecies competition among the preys is considered. The controls are pesticides or insecticides that act on the preys or the predators. The optimal control is synthesized and the dependence of the response time on the parameters of the problem is analysed. The time-optimal control problem for a predator–prey system without intraspecies competition has been previously studied for the Lotka–Volterra model [1] and for the Mono model [2].

1. STATEMENT OF THE PROBLEM

CONSIDER a controlled system modelling the interaction of two populations with intraspecies competition [3]

$$X_1^{\boldsymbol{\cdot}}(\tau) = (a_1 - a_2 Y_1(\tau) - a_5 X_1(\tau) - a_6 u_1(\tau)) X_1(\tau)$$
$$Y_1^{\boldsymbol{\cdot}}(\tau) = (a_3 X_1(\tau) - a_4 - a_7 Y_1(\tau) - a_8 u_1(\tau)) Y_1(\tau)$$

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